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ARONSZAJN TREES AND PARTITIONS

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ABSTRACT
A kind of κ^+-Aronszajn tree is used to construct some strong negative partition relations on κ^+.

A tree T is λ-Aronszajn iff T has height λ, has levels of size $<\lambda$, but has no chains of size λ. If T is the union of $\leqq\kappa$ antichains then T is called κ-special. In this note (without loss of generality) our trees will always have the property that different points of the same limit level have different sets of predecessors. T is called a θ-Cantor tree iff height$(T)=\gamma+1$ for some cardinal $\gamma\leqq\theta$, the first γ levels of T have size $\leqq\theta$, but the last level of T has size $>\theta$. In this note we give some applications (announced in [11]) of the following two results from [10; §4] and [12; §8], respectively.

THEOREM 1. *Assume $\kappa\geqq\omega$ and $\square_\kappa$. Then there is a κ-special κ^+-Aronszajn tree with no λ-Aronszajn nor θ-Cantor subtrees for any regular $\lambda\neq\kappa^+$ and infinite θ.*

THEOREM 2. *Assume* $\operatorname{cf}\kappa=\omega$ *and $\square_\kappa$. Then there is an order type ϕ of size κ^+ with a dense set of size κ such that every $\phi'\leqq\phi$ of size $\leqq\kappa$ is the union of countably many well ordered subtypes.*

Let us say a few words about the proofs of Theorems 1 and 2. The tree T of Theorem 1 consists of certain well-ordered subsets of Q_κ, where Q_κ is the set of all functions from ω into κ which are eventually equal to 0. Each element of T has a maximal element. The construction of T is as canonical as possible (using $\square_\kappa$) and this canonicity is used in showing that T has no λ-Aronszajn nor θ-Cantor subtrees of smaller heights. The type ϕ of Theorem 2 is equal to

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$\mathrm{tp}(L, <)$ where L is a certain subset of ${}^{\omega}\kappa$ of size κ^+ and $<$ is the lexicographical ordering of ${}^{\omega}\kappa$. The set L is constructed inductively together with maps which guarantee that any $K \subseteq L$ of size $\leqq \kappa$ is the union of $\leqq \aleph_0$ well-ordered subsets.

The tree T of Theorem 1 cannot be constructed in ZFC or ZFC + GCH ([10; §4]). The assumption $\mathrm{cf}\,\kappa = \omega$ in Theorem 2 is easily seen to be necessary. It is also known that such an order type cannot be constructed in ZFC + GCH ([1]).

Almost all of our applications of Theorem 1 and 2 are motivated by problems from the Erdös–Hajnal list [3; Problems 17, 19, 62] (see also [4]). We shall use the tree of Theorem 1, among other things, in stepping-up negative square-bracket partition relations. This stepping-up seems to be rather different from the classical one ([5]) which uses trees of height κ with $> \kappa$ cofinal branches.

We begin with two applications of Theorem 2. Let ϕ be the order type which satisfies the conclusion of this theorem. Since $d(\phi) = \kappa$, ϕ can be represented as $\mathrm{tp}(\mathcal{F}, \subseteq)$ where $\mathcal{F}$ is a chain of subsets of κ. Using the third property of ϕ it is easily seen that $\mathcal{F}$ is a κ-Kurepa family, i.e., $|\mathcal{F}| > \kappa$, but $|\{F \cap X : F \in \mathcal{F}\}| \leqq |X|$ for any infinite $X \subseteq \kappa$ of size $< \kappa$. Thus we have the following result which should be compared with [3; Problem 19E]:

THEOREM 3. *If* $\mathrm{cf}\,\kappa = \omega$, *then* $\square_\kappa$ *implies* $\mathrm{KH}_{\kappa,\kappa}$.

Concerning this result, let us note that K. Prikry ([8]) has previously shown that $V = L$ implies $\mathrm{KH}_{\kappa,\kappa}$ when $\mathrm{cf}\,\kappa = \omega$. It is unknown whether GCH implies $\mathrm{KH}_{\kappa,\kappa}$ for $\kappa = \aleph_\omega$.

Again let $\phi = \mathrm{tp}(\kappa^+, <)$ be the type of Theorem 2. Let $G \subseteq [\kappa^+]^2$ be the Sierpiński graph on κ^+ generated by ϕ, i.e., $\{\alpha, \beta\} \in G$ iff $\alpha < \beta$ and $\alpha < \beta$. Clearly, G satisfies the following theorem which should be compared with [5; Problem 7], where $[A]^\lambda$ denotes the set $\{B \subseteq A : |B| = \lambda\}$.

THEOREM 4. *Assume* $\mathrm{cf}\,\kappa = \omega$ *and* $\square_\kappa$. *Then there is a graph* $G \subseteq [\kappa^+]^2$ *such that* G *has no complete subgraph of size* κ^+, *but if* $\omega < \mathrm{cf}\,\lambda \leqq \lambda \leqq \kappa$ *and if* $A \in [\kappa^+]^\lambda$, *then* $[B]^2 \subseteq G$ *for some* $B \in [A]^\lambda$.

From now on we shall be using the κ^+-Aronszajn tree of Theorem 1. It is easily seen that if there is any such a tree then there is one which is an initial segment of ${}^{\kappa^+}2$. So from now on we shall assume that any κ^+-Aronszajn tree with no Aronszajn nor Cantor subtrees of smaller heights we are working with is an initial segment of ${}^{\kappa^+}2$. Such a tree, in short, will be called a "nice" κ^+-Aronszajn tree. So if T is a nice κ^+-Aronszajn tree then for every $s, t \in T$ we can define $s \wedge t$ to be the greatest lower bound of s and t. The property of nice κ^+-Aronszajn trees we shall use is the following.

LEMMA 1. *Let T be a nice κ^+-Aronszajn tree, and let $A \subseteq T$ have regular cardinality $\lambda \leqq \kappa$. Then there are sequences $\langle s_\alpha : \alpha < \lambda \rangle$ and $\langle t_\alpha : \alpha < \lambda \rangle$ from A and T, respectively such that $s_\alpha \wedge s_\beta = t_\alpha$ for all $\alpha < \beta < \lambda$.*

PROOF. Let S be the set of all s in T such that for all $i < 2$, $s^\frown i$ has an extension in A. Then S is a rooted at most 2-splitting subtree of T of size λ. By restricting S to its first λ levels we may assume that S has height $\leqq \lambda$. Levels of S must all have size $< \lambda$ since otherwise, by considering minimal λ-sized level of S, we get a θ-Cantor subtree of T. Since S cannot be λ-Aronszajn it must have a λ-branch which is just what is needed in Lemma 1.

Note that the conclusion of Lemma 1 implies $t_\alpha \subset t_\beta$ for $\alpha < \beta < \lambda$ and that the fact that T is κ^+-Aronszajn is not used. The conclusion of the lemma is, in fact, equivalent to saying T contains neither Aronszajn nor Cantor subtrees of smaller heights.

To mention our next application we need to define two partition symbols of Erdös, Hajnal and Rado ([5], [3]). The symbol $\kappa \to [\lambda_\xi]^r_{\xi<\theta}$ means that, for every $f : [\kappa]^r \to \theta$ there are $\xi < \theta$ and $A \in [\kappa]^{\lambda_\xi}$ such that $\xi \notin f''[A]^r$. The symbol $\kappa \to [\lambda]^r_{\delta,<\varepsilon}$ means that for every $f : [\kappa]^r \to \delta$, there exists $A \in [\kappa]^\lambda$ such that $|f''[A]^r| < \varepsilon$. Finally, $\kappa \to [\lambda]^r_{\delta,\varepsilon}$ means $\kappa \to [\lambda]^r_{\delta,<\varepsilon^+}$.

THEOREM 5. *$\square_\kappa$ implies $\kappa^+ \not\to [\lambda]^2_{\kappa,<\lambda}$ for every $\lambda \leqq \kappa$.*

PROOF. Let T be a nice κ-special κ^+-Aronszajn tree. It suffices to find a partition $f : [T]^2 \to \kappa$ which witnesses $\kappa^+ \not\to [\lambda]^2_{\kappa,<\lambda}$ for all regular $\lambda \leqq \kappa$. Let $\sigma : T \to \kappa$ be a specializing map, i.e., $\sigma(s) \neq \sigma(t)$ for $s \subset t$. Let

$$f(\{s,t\}) = \sigma(s \wedge t).$$

Using Lemma 1 it follows directly that f is as required.

Note that the partition f witnesses $\kappa^+ \not\to [\lambda]^2_{\kappa,<\lambda}$ for all $\lambda \leqq \kappa$, simultaneously. This should be compared with [5; p. 156] and [3; Problem 19].

In [2] Chang proved that $\forall n < \omega \, \mathrm{KH}_{\aleph_n,\aleph_n}$ implies $\aleph_\omega \not\to [\aleph_\omega]^{<\omega}_{\aleph_\omega}$, i.e., there is an $\aleph_\omega$-Jónsson algebra. Using Aronszajn trees instead of Kurepa trees in Chang's proof we get the following.

THEOREM 6. *Assume $\square_{\omega_n}$ holds for all $n < \omega$. Then $\aleph_\omega \not\to [\aleph_\omega]^{<\omega}_{\aleph_\omega}$.*

Now we are going to use nice κ-special κ^+-Aronszajn trees to step-up negative square bracket partition relations, thereby commenting on Problem 17 from [3]. For κ and λ cardinals, $\kappa \dot{+} \lambda$ denotes the cardinal sum of κ and λ. We shall need the following lemma which is proved by an easy induction on r.

LEMMA 2. *Let T be a nice κ^+-Aronszajn tree, let $1 \leqq r < \omega$ and let $A \in [T]^r$. Then $\rho(A) = \{s \wedge t : s, t \in A$ and $s \neq t\}$ is a subset of T of size $\leqq r-1$.*

THEOREM 7. *Assume $\square_\kappa$. Let $1 \leqq r < \omega$ and λ_ξ, $\xi < \theta$ be cardinals $\leqq \kappa$ such that all of them except possibly λ_0 are regular and infinite. Then*

$$\kappa \not\to [\lambda_\xi]^r_{\xi<\theta} \quad \textit{implies} \quad \kappa^+ \not\to [\lambda_\xi \dot{+} 1]^{r+1}_{\xi<\theta}.$$

PROOF. Let $g : [\kappa]^r \to \theta$ be a witness of $\kappa \not\to [\lambda_\xi]^r_{\xi<\theta}$. Let T be a nice κ-special κ^+-Aronszajn tree. We shall find a partition $f : [T]^{r+1} \to \theta$ which witnesses $\kappa^+ \not\to [\lambda_\xi \dot{+} 1]^{r+1}_{\xi<\theta}$. Let $\sigma : T \to \kappa$ be a specializing map and let $\prec$ be a well-ordering of T. Let

$$f(\{t_0, \ldots, t_r\}_\prec) = \begin{cases} \xi & \text{if } t_0 \wedge t_1 \subset t_1 \wedge t_2 \subset \cdots \subset t_{r-1} \wedge t_r \quad \text{and} \quad g(\sigma''\rho\{t_0, \ldots, t_r\}) = \xi, \\ 0 & \text{otherwise.} \end{cases}$$

Assume first by way of contradiction that for some $A \subseteq T$, $|A| = \lambda_0 \dot{+} 1$ and $0 \not\in f''[A]^{r+1}$. Let $\langle t_i : i < \lambda_0 \dot{+} 1 \rangle$ be the $\prec$-increasing enumeration of A. Then by the definition of f, we have $t_i \wedge t_{i+1} \subset t_{i+1} \wedge t_{i+2}$ for every $i < \lambda_0 - 1$. Let $s_i = t_i \wedge t_{i+1}$ for $i < \lambda_0$. Then $C = \{s_i : i < \lambda_0\}$ is a chain of T of size λ_0, and so $B = \sigma''C$ is also of size λ_0. Note that every element of $[B]^r$ is equal to $\sigma''\rho(X)$ for some $X \in [A]^{r+1}$. Hence $0 \not\in g''[B]^r$, a contradiction.

Assume now that for some $0 < \xi < \theta$ and $A \in [T]^{\lambda_\xi}$, we have $\xi \not\in f''[A]^{r+1}$. By Lemma 1 there exist $\prec$-increasing sequences $\langle s_\alpha : \alpha < \lambda_\xi \rangle$ and $\langle t_\alpha : \alpha < \lambda_\xi \rangle$ of A and T, respectively so that $s_\alpha \wedge s_\beta = t_\alpha$ for every $\alpha < \beta < \lambda_\xi$. Let $B = \sigma''\{t_\alpha : \alpha < \lambda_\xi\}$. Then it is easily seen that every element of $[B]^r$ is equal to $\sigma''\rho(X)$ for some $X \in [A]^{r+1}$. Hence $\xi \not\in g''[B]^r$, which is a contradiction since $|B| = \lambda_\xi$ follows from the facts that $\{t_\alpha : \alpha < \lambda_\xi\}$ is a chain and σ is a specializing map. This completes the proof.

Using the above methods one can also step-up some special properties of partitions of $[\kappa]^r$. So we can also get analogues of Theorems 2.6 and 2.7 of [10]. Let us now give a typical application of Theorem 7.

THEOREM 8. $(V = L)$. *Let κ be a regular non-weakly compact cardinal. Then for every $n < \omega$ and $3 \leqq r < \omega$*

$$\kappa^{(n)+} \not\to [\kappa]^{n+2}_\kappa \quad \textit{and} \quad \kappa^{(n)+} \not\to [r+n+1, (\kappa)_\kappa]^{r+n}.$$

PROOF. $\kappa^{(n)+} \not\to [\kappa]^{n+2}_\kappa$ is obtained by stepping-up the relation $\kappa \not\to [\kappa]^2_\kappa$ which holds in L by results of Shore [9] and Jensen [7]. The second relation is obtained by stepping-up the relation $\kappa \not\to [r+1, (\kappa)_\kappa]^r$ which holds in L by the following proposition and a result of Jensen [7] which says that for every regular non-weakly compact cardinal κ there is a κ-Suslin tree.

THEOREM 9. *Assume κ is regular and there is a κ-Suslin tree. Then*

$$\kappa \not\rightarrow [r+1,(\kappa)_\kappa]^r \qquad \textit{for every } 3 \leqq r < \omega.$$

PROOF. Let $T = \langle \kappa, \leqq_T \rangle$ be a κ-Suslin tree so that $\alpha <_T \beta$ implies $\alpha < \beta$, and so that for every $\alpha \in T$, the set S_α of all immediate successors of α in T has order type $\geqq \alpha$. Let $\pi_\alpha : S_\alpha \rightarrow \mathrm{On}$ be the collapsing map. Let $\{\alpha_1, \ldots, \alpha_r\}_<$ be given. If for some $\beta \in T$, $\alpha_1 <_T \beta$ we have $\beta = \alpha_i \wedge \alpha_j \neq \alpha_i, \alpha_j$ for all $2 \leqq i \neq j \leqq r$, then we let

$$f(\{\alpha_1, \ldots, \alpha_r\}) = \pi_{\alpha_1}(\gamma), \qquad \text{where } \{\gamma\} = \{\xi \in T : \xi \leqq_T \beta\} \cap S_{\alpha_1}.$$

Otherwise, let $f(\{\alpha_1, \ldots, \alpha_r\}) = 0$.

Since T is κ-Suslin any $A \in [T]^\kappa$ is dense in a κ-sized cone of T, so for any $0 < \xi < \kappa$ there is an $X \in [A]^r$ such that $f(X) = \xi$. But given any set $B \subseteq T$ of size $r+1$ there must be an element X of $[B]^r$ such that $f(X) = 0$. This shows that f witnesses $\kappa \not\rightarrow [r+1,(\kappa)_\kappa]^r$.

Concerning Theorems 8 and 9 let us mention that Hajnal [6] has shown that $\kappa \rightarrow (r+1,\kappa)^r$ for some $r \geqq 3$ is already enough to imply that κ is a weakly compact cardinal.

Let us conclude this paper with a remark concerning the assumption $\square_\kappa$ in Theorems 5 and 7. Chang's Conjecture (CC) is the assertion that any first order structure of the form $\langle \omega_2, \omega_1, \ldots \rangle$ for countable language has an elementary substructure $\langle B, B \cap \omega_1, \ldots \rangle$ such that $|B| = \aleph_1$ and $|B \cap \omega_1| = \aleph_0$. It is well-known that CC is equivalent to $\aleph_2 \rightarrow [\aleph_1]^2_{\aleph_1, \aleph_0}$. So by Theorem 5 we have that CC implies $\neg \square_{\omega_1}$. On the other hand a well-known result of J. Silver says that CC is consistent with ZFC + GCH. Thus the conclusion of Theorem 5 does not follow from GCH. Clearly CC implies $\aleph_2 \rightarrow [\aleph_1]^3_{\aleph_1}$, hence the conclusion of Theorem 7 is not provable in ZFC + GCH since CH implies $\aleph_1 \not\rightarrow [\aleph_1]^2_{\aleph_1}$ ([5]). An unpublished result of R. Laver says that, in fact, $\aleph_2 \rightarrow [\aleph_1]^3_{\aleph_1}$ is equivalent to CC if CH holds.

Note that $\kappa \rightarrow [\lambda_\xi]^r_{\xi<\theta}$ makes sense even if κ and λ_ξ's are just ordinal numbers not necessarily initial. The proof of Theorem 7 shows clearly that we can restate this result in the following form: Assume κ is a cardinal for which $\square_\kappa$ holds. Let $1 \leqq r < \omega$ and let λ_ξ, $\xi < \theta$ be *ordinals* $\leqq \kappa$ such that all of them except possibly λ_0 are regular and infinite. Then

$$\kappa \not\rightarrow [\lambda_\xi]^r_{\xi<\theta} \quad \text{implies} \quad \kappa^+ \not\rightarrow [\lambda_0 + 1, (\lambda_\xi \dotplus 1)_{1 \leqq \xi < \theta}]^{r+1}$$

where $\lambda_0 + 1$ is the ordinal sum of λ_0 and 1. Thus, for example, if we assume $\square_{\omega_n}$

for all $n < \omega$, then we can step up the well known CH-consequence

$$\omega_1 \not\to [\omega + 2, (\omega_1)_{\aleph_1}]^2$$

of A. Hajnal to obtain

$$\omega_{n+1} \not\to [\omega + n + 2, (\omega_1)_{\aleph_1}]^{n+2} \qquad \text{for all } n < \omega.$$

This solves a recent problem of A. Hajnal and P. Komjáth.

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